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1997 J. Phys. A: Math. Gen. 30 7895

(http://iopscience.iop.org/0305-4470/30/22/025)

J. Phys. A: Math. Gen. **30** (1997) 7895–7917. Printed in the UK PII: S0305-4470(97)79880-9

Quantization and scattering of massive scalar fields on exterior domains

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Received 27 November 1996

Abstract. A quantum theory is developed for the scattering of massive scalar fields on a class of non-globally hyperbolic spacetimes represented by foliations of Minkowski spacetime with a fixed compact set removed from each Cauchy surface. The field is restricted to the exterior of this set (exterior domain). At the classical level, the boundary value problem is recast as an abstract Cauchy problem in a Hilbert space, and the field solution is obtained as a unitary mapping of Cauchy data. The scattering theory is treated using a two Hilbert space approach, and the wave operators are constructed and shown to be asymptotically complete. At the quantum level, a field operator is constructed yielding a representation of the CCRs on a Fock space. Representations for the 'in' and 'out' asymptotic fields are developed, and a scattering operator is constructed and shown to be unitarily implementable.

1. Introduction

The quantization of linear quantum fields on globally hyperbolic spacetimes is well understood on a mathematical level, and considerable progress has been made in the development of rigorous scattering theories for such spacetimes [1–20]. Recall that a spacetime $(\mathcal{M}, g)$ is globally hyperbolic if the strong causality assumption holds on $\mathcal{M}$, and if for any two point $p, q \in \mathcal{M}$, $J^+(p) \cap J^-(q)$ is compact and contained in $\mathcal{M}$ ($J^{\pm}(p)$ is the causal future/past of p) [21]. Apparently few rigorous results exist for field quantization and scattering on non-globally hyperbolic spacetimes [22, 23]. Such spacetimes are not without interest as exemplified by the Casmir effect in which the field is confined to a spacial region between two parallel boundaries [1].

In this article, we develop a quantum scattering theory for massive scalar fields on a class of non-globally hyperbolic spacetimes. Specifically, we study field quantization on foliations of Minkowski spacetime with a fixed compact set $\Lambda \subset \mathbb{R}^3$ removed from each Cauchy surface, i.e.

$$\mathcal{M} \approx \mathbb{R} \times \Omega$$

where $\Omega \equiv \mathbb{R}^3 \backslash \Lambda$. The field is restricted to the 'exterior domain' Ω, and takes on a prescribed boundary condition on $\partial\Omega$ (boundary of Ω). Note that the absence of Λ renders $\mathcal{M}$ non-globally hyperbolic because $J^+(p) \cap J^-(q)$ is not always compact.

The analysis divides into the classical and quantum problems. For the classical problem, we reformulate the boundary value problem as an operator equation in a Hilbert space and obtain a field solution as a unitary mapping of Cauchy data. The classical scattering theory is treated using a two Hilbert space approach. Wave operators are constructed and shown to be asymptotically complete, thus giving rise to a scattering operator.

0305-4470/97/227895+23$19.50

For the quantum problem, we construct a field operator on a Fock space thereby obtaining a representation of the CCRs. 'In' and 'out' asymptotic fields are defined and the corresponding Weyl algebras are constructed. We define vacuum states for these algebras, and then use a Gel'fand–Naimark–Segal (GNS) construction to obtain Hilbert space representations for these states. Finally, we obtain representations of both algebras on the same Hilbert space, and define a mapping from one algebra to the other. We show that this mapping is unitarily implementable and obtain an explicit representation for the quantum scattering operator.

2. The classical problem

Consider the scattering of a scalar field off an obstacle in the form of a compact subset $\Lambda \subset \mathbb{R}^3$. We choose a reference frame at rest with respect to Λ, and seek a solution to

$$\Box\phi + m^2\phi = 0 \qquad m \in (0, \infty) \tag{2.1}$$

on the (connected) exterior domain $\Omega \equiv \mathbb{R}^3 \backslash \Lambda$ which has a C^∞ boundary $\partial\Omega$ of measure zero. For the problem to be well posed we specify the Dirichlet boundary condition on $\partial\Omega$. Since we do not specify any particular shape for Λ, we need to employ abstract methods. To this end, we reformulate this classical boundary value problem as an abstract Cauchy problem in a Hilbert space. This problem has been studied extensively for the wave equation ($m = 0$) [24].

2.1. The Cauchy problem

Let $C_0^\infty(\Omega)$ denote the infinitely differentiable, complex-valued functions with compact support on Ω, and let $\overset{\circ}{\mathcal{H}}_m(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|_1^2 \equiv \sum_{|\alpha|\leqslant m} \int_\Omega |D^\alpha f|^2 \,\mathrm{d}x$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}$. The corresponding real-valued spaces are $C_0^\infty(\Omega, \mathbb{R})$ and $\overset{\circ}{\mathcal{H}}_m(\Omega, \mathbb{R})$ (functions and spaces are complex unless indicated otherwise). There are also the Sobolev spaces

$$\mathcal{H}_m(\Omega) = \{f : D^\alpha f \in L^2(\Omega), |\alpha| \leqslant m\}$$

and it is well known that $\overset{\circ}{\mathcal{H}}_m(\Omega) \subset \mathcal{H}_m(\Omega)$ (proper subspace). Note that $\overset{\circ}{\mathcal{H}}_1(\mathbb{R}^3) = \mathcal{H}_1(\mathbb{R}^3)$.

In this section, we work primarily with $\overset{\circ}{\mathcal{H}}_1(\Omega)$ whose elements satisfy the Dirichlet condition on $\partial\Omega$ in a generalized sense. The goal is to obtain an $\overset{\circ}{\mathcal{H}}_1(\Omega)$-valued solution to (2.1). This equation can be rewritten as a first-order system,

$$\frac{\partial}{\partial t}\begin{pmatrix}\phi(t)\\ \pi(t)\end{pmatrix} = -\mathrm{i}H\begin{pmatrix}\phi(t)\\ \pi(t)\end{pmatrix} \tag{2.2}$$

where $\pi(t) = \partial_t\phi(t)$,

$$H = \mathrm{i}\begin{pmatrix}0 & I\\ -B^2 & 0\end{pmatrix} \tag{2.3}$$

and

$$B^2 \equiv -\triangle_{\mathrm{D}}^{\Omega} + m^2 \tag{2.4}$$

where $-\triangle_{\mathrm{D}}^{\Omega}$ is the Dirichlet Laplacian (Friedrich's extension) on Ω, i.e. it is the unique self-adjoint operator on $L^2(\Omega, \mathrm{d}x)$ whose quadratic form is the closure of

$$q(f, g) = \sum_{|\alpha|=1} \int_{\Omega} (\overline{D^{\alpha} f})(D^{\alpha} g)\, \mathrm{d}x$$

with domain $C_0^{\infty}(\Omega)$ [25]. Note that B^2 and $-\triangle_{\mathrm{D}}^{\Omega}$ have the same form domain, $Q(B^2) = Q(-\triangle_{\mathrm{D}}^{\Omega}) = \mathring{\mathcal{H}}_1(\Omega)$. Let $B = \sqrt{-\triangle_{\mathrm{D}}^{\Omega} + m^2}$ denote the square root of the strictly positive self-adjoint operator B^2, and let $\mathcal{H}(\Omega)$ denote the Hilbert space

$$\mathcal{H}(\Omega) \equiv D(B) \oplus L^2(\Omega) \tag{2.5}$$

with norm

$$\|F\|^2_{\mathcal{H}(\Omega)} = \|f_1\|^2_B + \|f_2\|^2_{L^2(\Omega)}$$

where $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, and

$$\|f\|^2_B \equiv \langle Bf, Bf \rangle_{L^2(\Omega)}.$$

We note that $D(B) = D(\sqrt{-\triangle_{\mathrm{D}}^{\Omega}}) = Q(-\triangle_{\mathrm{D}}^{\Omega})$. It is well known that H is self-adjoint on $\mathcal{H}(\Omega)$ with $D(H) = D(B^2) \oplus D(B)$ [26]. From Stone's theorem we have a group of unitary operators $U(t, s) \equiv \exp(-\mathrm{i}H(t - s))$ and from the functional calculus [26],

$$U(t, 0) = \begin{pmatrix} \cos(Bt) & B^{-1}\sin(Bt) \\ -B\sin(Bt) & \cos(Bt) \end{pmatrix}. \tag{2.6}$$

Thus, we obtain an $\mathcal{H}(\Omega)$-valued solution to the Cauchy problem of the form

$$F(t) = U(t, 0)F$$

where

$$\partial_t F(t) = -\mathrm{i}HF(t)$$

and $F \in D(H)$ are the initial data. Note that the first component $f_1(t, \cdot) \in \mathring{\mathcal{H}}_1(\Omega)$ as desired.

Remark 1. If the initial data are smooth, i.e. $F \in C_0^{\infty}(\Omega) \times C_0^{\infty}(\Omega)$, then the solution $F(t)$ is C^{∞} in t and x. This is not an obvious fact, but the argument for it is standard. First, for smooth F we have $\frac{\mathrm{d}^n}{\mathrm{d}t^n} F(t) = (-\mathrm{i}H)^n F(t) = U(t, 0)(-\mathrm{i}H)^n F \in \mathcal{H}(\Omega)$ which implies that $f_i(t, \cdot)$ is infinitely often, strongly differentiable in t with respect to $\|\cdot\|_{L^2(\Omega)}$ ($f_i(t, \cdot)$ is the ith component of $F(t)$). It is easy to show that $f_i(t, \cdot)$ is locally an element of $L^2(\mathbb{R}, L^2(\Omega)) = L^2(\mathbb{R} \times \Omega)$ (see section II.4 of [27]). Moreover, the L^2 derivatives $\frac{\mathrm{d}^n}{\mathrm{d}t^n} f_i(t, \cdot)$ equal the distributional time derivatives. We also have that $(-\triangle_{\mathrm{D}}^{\Omega})^n f_i(t, \cdot) \in L^2(\Omega)$, for any non-negative integer n. Therefore, from Sobolev's lemma, f_i can be uniquely identified with an element of $C^{\infty}(\mathbb{R} \times \Omega)$ [28].

We denote the interacting system by the triplet $\{\mathcal{H}(\Omega), H, U(t, 0)\}$, and the corresponding free system by $\{\mathcal{H}_0(\mathbb{R}^3), H_0, U_0(t, 0)\}$ for which $\Omega = \mathbb{R}^3$ and $U_0(t, s) \equiv \exp(-\mathrm{i}H_0(t - s))$, with

$$H_0 = \mathrm{i} \begin{pmatrix} 0 & I \\ -B_0^2 & 0 \end{pmatrix}$$

where $D(H_0) = D(B_0^2) \oplus D(B_0)$, $B_0^2 = -\triangle + m^2$, and $-\triangle$ is the Laplacian on $L^2(\mathbb{R}^3)$. Note,

$$\mathcal{H}_0(\mathbb{R}^3) \equiv D(B_0) \oplus L^2(\mathbb{R}^3)$$

where $D(B_0) = \mathcal{H}_1(\mathbb{R}^3)$, and

$$\langle F, G\rangle_{\mathcal{H}_0(\mathbb{R}^3)} \equiv \langle f_1, g_1\rangle_{B_0} + \langle f_2, g_2\rangle_{L^2(\mathbb{R}^3)} \tag{2.7}$$

with

$$\langle f_1, g_1\rangle_{B_0} \equiv \langle B_0 f_1, B_0 g_1\rangle_{L^2(\mathbb{R}^3)}.$$

For the free dynamics, it is well known that if the initial data F are smooth then the solution $F(t) = U(t, 0)F$ is a smooth classical solution and that [29]

$$\sup_x |f_1(t, x)| < \frac{c}{|t|^{\frac{3}{2}}} \qquad |t| \geqslant 1 \tag{2.8}$$

where $F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$. It readily follows that

$$\|f_1(t)\|_{L^2(B)} < c\frac{Vol(B)}{|t|^{\frac{3}{2}}} \tag{2.9}$$

for any bounded subset $B \subset \mathbb{R}^3$. Moreover, $f_2(t)$ is also a smooth solution and the same estimate applies to it.

2.2. Symplectic structure

In this section we describe free and interacting dynamical systems and corresponding symplectic structures that are needed for the quantum problem. We start with the free dynamics, let

$$\mathcal{H}_0(\mathbb{R}^3, \mathbb{R}) \equiv \mathcal{H}_1(\mathbb{R}^3, \mathbb{R}) \oplus L^2(\mathbb{R}^3, \mathbb{R}). \tag{2.10}$$

We prove the following preliminary result.

Proposition 1. The free evolution operator maps real-valued data to real-valued data, i.e. $U_0(t, 0) : \mathcal{H}_0(\mathbb{R}^3, \mathbb{R}) \to \mathcal{H}_0(\mathbb{R}^3, \mathbb{R})$.

Proof. Note, $H_0 = \mathrm{i}\tilde{H}_0$ where $\tilde{H}_0$ is a real operator. Since H_0 is self-adjoint, $\tilde{H}_0$ is skew-adjoint, and so is its restriction (also denoted $\tilde{H}_0$) to $\mathcal{H}_0(\mathbb{R}^3, \mathbb{R})$ with

$$D(\tilde{H}_0) \equiv [D(B_0^2) \cap \mathcal{H}_1(\mathbb{R}^3, \mathbb{R})] \oplus [D(B_0) \cap L^2(\mathbb{R}^3, \mathbb{R})].$$

Thus, from Stone's theorem, $\tilde{H}_0$ generates a unitary group $\tilde{U}_0(t, 0) : \mathcal{H}_0(\mathbb{R}^3, \mathbb{R}) \to \mathcal{H}_0(\mathbb{R}^3, \mathbb{R})$, which is just the restriction of $U(t, 0)$ to $\mathcal{H}_0(\mathbb{R}^3, \mathbb{R})$. □

For the free dynamics there is a conserved current ($\partial^\mu j_\mu = 0$) where

$$j_\mu = \phi_1 \partial_\mu \phi_2 - (\partial_\mu \phi_1)\phi_2$$

and ϕ_1, ϕ_2 are smooth real-valued solutions of (2.1). The time component j_0 gives rise to the form

$$\int_{\mathbb{R}^3} (\phi_1 \partial_t \phi_2 - (\partial_t \phi_1)\phi_2)\, \mathrm{d}x \tag{2.11}$$

which is independent of time. We introduce the symplectic form (skew symmetric and non-degenerate),

$$\sigma_{\mathbb{R}^3}(F, G) \equiv \int_{\mathbb{R}^3} (f_1 g_2 - f_2 g_1)\, \mathrm{d}x$$

and write the invariance of (2.11) as follows,

$$\sigma_{\mathbb{R}^3}(U_0(t,0)F, U_0(t,0)G) = \sigma_{\mathbb{R}^3}(F,G) \tag{2.12}$$

for smooth F and G ($F(t) = U_0(t,0)F \in C^\infty(\mathbb{R}^4)$ for smooth F). Since

$$|\sigma_{\mathbb{R}^3}(F,G)| \leqslant c\|F\|_{\mathcal{H}_0(\mathbb{R}^3,\mathbb{R})}\|G\|_{\mathcal{H}_0(\mathbb{R}^3,\mathbb{R})}$$

for some constant c, (2.12) holds on all of $\mathcal{H}_0(\mathbb{R}^3,\mathbb{R})$.

Finally, it is well known that the free dynamics can be diagonalized by use of the transform [26]

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} B_0^{\frac{1}{2}} & \mathrm{i}B_0^{-\frac{1}{2}} \\ B_0^{\frac{1}{2}} & -\mathrm{i}B_0^{-\frac{1}{2}} \end{pmatrix}. \tag{2.13}$$

Specifically, $h_0 \equiv TH_0T^{-1}$ or

$$h_0 = -\begin{pmatrix} -B_0 & 0 \\ 0 & B_0 \end{pmatrix}$$

which is self-adjoint on $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ with domain $D(h_0) = D(B_0) \oplus D(B_0)$. In terms of the new variables we have $a(t) = \exp(-\mathrm{i}h_0t)a(0)$ where

$$a(t) = \begin{pmatrix} a^+(t) \\ a^-(t) \end{pmatrix}$$

with components $a^\pm(t) = \exp(\mp\mathrm{i}B_0t)a^\pm(0)$ which are the positive/negative frequency components. In terms of the original variables we have

$$a(t) = TU_0(t,0)F$$

and thus the operator

$$K_0F \equiv \frac{1}{\sqrt{2}}[B_0^{\frac{1}{2}}f_1 + iB_0^{-\frac{1}{2}}f_2] \tag{2.14}$$

maps the field solution to its positive frequency component. Note, $D(B_0) \subset D(B_0^{\frac{1}{2}})$ and $L^2(\mathbb{R}^3,\mathbb{R}) \subset D(B_0^{-\frac{1}{2}})$ ($B_0^{-\frac{1}{2}}$ is bounded), therefore,

$$\mathcal{H}_0(\mathbb{R}^3,\mathbb{R}) \subset D(K_0). \tag{2.15}$$

Also, since $Ran(B_0^{\frac{1}{2}}) = D(B_0^{-\frac{1}{2}}) = L^2(\mathbb{R}^3)$ and $Ran(B_0^{-\frac{1}{2}}) = D(B_0^{\frac{1}{2}})$ is dense in $L^2(\mathbb{R}^3)$ we have that K_0 maps $\mathcal{H}_0(\mathbb{R}^3,\mathbb{R})$ onto a dense set in $L^2(\mathbb{R}^3)$. Moreover K_0, is symplectic, i.e.

$$\sigma_{\mathbb{R}^3}(F,G) = 2\,\mathrm{Im}\langle K_0F, K_0G\rangle_{L^2(\mathbb{R}^3)}.$$

This operator plays a key role in the development of the quantum theory.

Now consider the interacting system, let

$$\mathcal{H}(\Omega,\mathbb{R}) \equiv \mathring{\mathcal{H}}_1(\Omega,\mathbb{R}) \oplus L^2(\Omega,\mathbb{R}). \tag{2.16}$$

Proposition 2. The interacting evolution operator maps real-valued data to real-valued data, i.e. $U(t,0) : \mathcal{H}(\Omega,\mathbb{R}) \to \mathcal{H}(\Omega,\mathbb{R})$.

Proof. This proof is the same as in proposition 1 with H_0 replaced H, $\mathbb{R}^3$ replaced by Ω etc. $\square$

We define the symplectic form

$$\sigma_\Omega(F, G) \equiv \int_\Omega (f_1 g_2 - f_2 g_1)\,\mathrm{d}x$$

and prove

Theorem 2.1. Let $F, G \in \mathcal{H}(\Omega, \mathbb{R})$ then

$$\sigma_\Omega(U(t,s)F, U(t,s)G) = \sigma_\Omega(F, G).$$

Proof. First, note that $H = \mathrm{i}\tilde{H}$ where $\tilde{H}$ is a skew-adjoint real operator on $\mathcal{H}(\Omega, \mathbb{R})$ with

$$D(\tilde{H}) \equiv [D(B^2) \cap \mathring{\mathcal{H}}_1(\Omega, \mathbb{R})] \oplus [D(B) \cap L^2(\Omega, \mathbb{R})]$$

it readily follows that

$$\sigma_\Omega(\tilde{H}F, G) = -\sigma_\Omega(F, \tilde{H}G) \tag{2.17}$$

for $F, G \in D(\tilde{H})$. Also, note that $\sigma_\Omega(F, G)$ is bounded, i.e.

$$|\sigma_\Omega(F, G)| \leqslant c\|F\|_{\mathcal{H}(\Omega,\mathbb{R})}\|G\|_{\mathcal{H}(\Omega,\mathbb{R})} \tag{2.18}$$

and hence continuous. Now, consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\Omega(F(t), G(t)) = \sigma_\Omega(\tilde{H}F(t), G(t)) + \sigma_\Omega(F(t), \tilde{H}G(t)) \tag{2.19}$$

where $F(t) = U(t,0)F$, $G(t) = U(t,0)G$ with $F, G \in D(\tilde{H})$. This follows from (2.18), and the fact that $U(t,0)$ is strongly differentiable on $D(\tilde{H})$. Next, apply (2.17) to (2.19) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\Omega(F(t), G(t)) = 0$$

which implies

$$\sigma_\Omega(F(t), G(t)) = \sigma_\Omega(F, G) \tag{2.20}$$

for all real valued $F, G \in D(\tilde{H})$. Finally, from (2.18) and the fact that $D(\tilde{H})$ is dense in $\mathcal{H}(\Omega, \mathbb{R})$ we have that (2.20) holds on all of $\mathcal{H}(\Omega, \mathbb{R})$. $\square$

In summary, we have free and interacting dynamical systems each of which consists of a real symplectic space, a symplectic form, and a one-parameter symplectic group. These are represented by the triplets $(\mathcal{H}_0(\mathbb{R}^3, \mathbb{R}), \sigma_\mathbb{R}(\cdot,\cdot), U_0(t,0))$ and $(\mathcal{H}(\Omega, \mathbb{R}), \sigma_\Omega(\cdot,\cdot), U(t,0))$, respectively. Moreover, there is a real-linear symplectic operator K_0 that maps free field solutions in $\mathcal{H}_0(\Omega, \mathbb{R})$ to their positive frequency components in $L^2(\mathbb{R}^3)$.

2.3. Scattering theory

We turn now to the classical scattering problem. The goal is to show that the scattered field behaves like a free field in the distant past and future. However, the free and interacting fields exist in different Hilbert spaces, and therefore, to compare them we use the two Hilbert space technique developed by Kato [30]. As a first step, we define an identification operator $J \in B(\mathcal{H}_0(\mathbb{R}^3), \mathcal{H}(\Omega))$ that maps free solutions to the interacting space.

Definition 1. An identification operator J is a smooth, real-valued, and bounded mapping $J : \mathcal{H}_0(\mathbb{R}^3) \to \mathcal{H}(\Omega)$ that maps to zero on some neighbourhood of $\partial\Omega$, and is the identity off a bounded set.

Remark 2. Note that any two identification operators differ only on a bounded set.

We expect the scattered solutions to look asymptotically free as $t \to \pm\infty$. We take this to mean that for $F_0 \in \mathcal{H}_0(\mathbb{R}^3)$ there is an $F \in \mathcal{H}(\Omega)$ such that

$$\lim_{t\to\pm\infty} \|e^{-iHt}F - Je^{-iH_0t}F_0\|_{\mathcal{H}(\Omega)} = 0 \tag{2.21}$$

where J is any identification operator. Since e^{-iHt} is unitary, (2.21) reduces to

$$\lim_{t\to\pm\infty} \|F - e^{iHt}Je^{-iH_0t}F_0\|_{\mathcal{H}(\Omega)} = 0$$

and the original problem amounts to proving the existence of strong limits,

$$W_\pm \equiv s - \lim_{t\to\pm\infty} W(t) \tag{2.22}$$

where

$$W(t) \equiv e^{iHt}Je^{-iH_0t}.$$

It turns out that $W_\pm$ exist and are complete, with

$$Ran(W_\pm) = P_{\mathrm{ac}}(H)\mathcal{H}(\Omega) \tag{2.23}$$

where $P_{\mathrm{ac}}(H)$ is the projection onto the absolutely continuous spectrum of H. However, the proof of this is tedious, and is given in appendix B.

Recall that $U(t,0)$, J and $U_0(t,0)$ are real operators, and therefore so are $W_\pm$, i.e.

$$W_\pm : \mathcal{H}_0(\mathbb{R}^3, \mathbb{R}) \to \mathcal{H}(\Omega, \mathbb{R}). \tag{2.24}$$

It readily follows that

$$\|W_\pm F\|_{\mathcal{H}(\Omega,\mathbb{R})} \leqslant c\|F\|_{\mathcal{H}_0(\mathbb{R}^3,\mathbb{R})} \tag{2.25}$$

for some constant c. Also note that

$$s - \lim_{t\to\pm\infty} W(t+\tau) = s - \lim_{t\to\pm\infty} W(t)$$

for any fixed τ, and therefore $W_\pm$ intertwine $U(t,0)$ and $U_0(t,0)$

$$U(t,0)W_\pm = W_\pm U_0(t,0) \tag{2.26}$$

which, from Stone's theorem, implies,

$$HW_\pm = W_\pm H_{0.} \tag{2.27}$$

From (2.26), and the fact that $W_\pm$ are complete, we have

$$W_\pm^{-1}U(t,0) = U_0(t,0)W_\pm^{-1}. \tag{2.28}$$

Finally, since $W_\pm$ are complete, we have the existence of the classical scattering operator

$$S_{\mathrm{cl}} \equiv W_+^{-1}W_-$$

and its inverse $S_{\mathrm{cl}}^{-1} = W_-^{-1}W_+$. Note that S_{cl} is a bijection from $\mathcal{H}_0(\mathbb{R}^3)$ onto itself. Moreover, from (2.26) and (2.28) we have

$$S_{\mathrm{cl}}U_0(t,0) = U_0(t,0)S_{\mathrm{cl}}. \tag{2.29}$$

The operator S_{cl} plays a key role in the development of the quantum scattering theory.

3. The quantum problem

The quantum theory begins with the construction of an interacting field operator on a Fock space. Let $\mathcal{H}^{(1)}$ be a complex Hilbert space, and let $\mathcal{F}_s(\mathcal{H}^{(1)})$ denote the Bose–Fock space over $\mathcal{H}^{(1)}$,

$$\mathcal{F}_s(\mathcal{H}^{(1)}) = \mathbb{C} \oplus \left(\bigoplus_{n=1}^{\infty} (\mathcal{H}^{(1)\otimes n})_s \right) \tag{3.1}$$

where the subscript s denotes the symmetric tensor product [31]. The vectors in this space can be represented by sequences of the form

$$\Psi = (\Psi^{(0)}, \Psi^{(1)}, \dots, \Psi^{(n)}, \dots) \tag{3.2}$$

where

$$\Psi^{(n)} = \frac{1}{\sqrt{n!}} \sum_{\pi} f_{\pi_1} \otimes f_{\pi_2} \otimes \cdots \otimes f_{\pi_n} \tag{3.3}$$

$f_1, \dots, f_n \in \mathcal{H}^{(1)}$, and the sum is over all permutations $\pi; (1, 2, \dots, n) \mapsto (\pi_1, \pi_2, \dots, \pi_n)$ of the indices. These vectors are invariant with respect to the interchange of their variables. The inner product on $\mathcal{F}_s(\mathcal{H}^{(1)})$ is induced by the inner product on $\mathcal{H}^{(1)}$,

$$\langle \Psi, \Psi \rangle_{\mathcal{F}_s} = |\Psi^{(0)}|^2 + \langle \Psi^{(1)}, \Psi^{(1)} \rangle_{\mathcal{H}^{(1)}} + \cdots + \langle \Psi^{(n)}, \Phi^{(n)} \rangle_{\mathcal{H}^{(n)}} + \cdots \tag{3.4}$$

where

$$\begin{aligned} \langle \Psi^{(n)}, \Psi^{(n)} \rangle_{\mathcal{H}^{(n)}} &= \frac{1}{n!} \left\langle \sum_{\pi'} f_{\pi'_1} \otimes \cdots \otimes f_{\pi'_n}, \sum_{\pi} f_{\pi_1} \otimes \cdots \otimes f_{\pi_n} \right\rangle_{\mathcal{H}^{(n)}} \\ &= \frac{1}{n!} \sum_{\pi'} \sum_{\pi} \prod_{i=1}^{n} \langle f_{\pi'_i}, f_{\pi_i} \rangle_{\mathcal{H}^{(1)}}. \end{aligned} \tag{3.5}$$

To complete a particle description we define creation and annihilation operators $a^*(\cdot)$ and $a(\cdot)$,

$$a^*(f)\Psi^{(n)} \equiv \frac{1}{\sqrt{(n+1)!}} \sum_{\pi} f_{\pi_0} \otimes f_{\pi_2} \otimes \cdots \otimes f_{\pi_n} \tag{3.6}$$

and

$$a(f)\Psi^{(n)} \equiv \frac{1}{\sqrt{(n-1)!}} \sum_{\pi} \langle f, f_{\pi_1} \rangle_{\mathcal{H}^{(1)}} f_{\pi_2} \otimes \cdots \otimes f_{\pi_n} \tag{3.7}$$

where, in (3.6), $f_0 = f$ and the permutations $\pi; (0, 1, \dots, n) \mapsto (\pi_0, \pi_1, \dots, \pi_n)$. Note that $a^*(\cdot)$ and $a(\cdot)$ are complex linear and antilinear in their arguments, respectively. Finally, we define the Fock spaces for the free and scattered fields,

$$\mathcal{F}_{\text{free}} \equiv \mathcal{F}_s(L^2(\mathbb{R}^3))$$

and

$$\mathcal{F}_{\text{scatt}} \equiv \mathcal{F}_s(L^2(\Omega)).$$

We denote the corresponding creation and annihilation operators by $a_0^*(\cdot)$, $a_0^*(\cdot)$,and $a^*(\cdot)$, $a(\cdot)$, respectively. There are also the finite particle subspaces $\mathcal{F}^0_{\text{free}}$ and $\mathcal{F}^0_{\text{scatt}}$ consisting of vectors in which all but a finite number of terms in (3.2) are zero.

We are now ready to construct the quantum fields. First, we construct the standard representation of the free field on $\mathcal{F}_{\text{free}}$. The time-zero free-field operator and its conjugate momentum are given by

$$\phi_0(f) \equiv \frac{1}{\sqrt{2}}[a_0^*(B_0^{-\frac{1}{2}} f) + a_0(B_0^{-\frac{1}{2}} f)]$$

and

$$\pi_0(g) \equiv \frac{\mathrm{i}}{\sqrt{2}}[a_0^*(B_0^{\frac{1}{2}} g) - a_0(B_0^{\frac{1}{2}} g)]$$

where the test functions are taken to be real-valued. To make use of the two-component classical theory, we introduce another field operator (at time zero)

$$\sigma_{\mathbb{R}^3}(\Phi^0, F) \equiv \phi_0(f_2) - \pi_0(f_1)$$

where $F \in \mathcal{H}_0(\mathbb{R}^3, \mathbb{R}) = \mathcal{H}_1(\mathbb{R}^3, \mathbb{R}) \oplus L^2(\mathbb{R}^3, \mathbb{R})$. This field can be rewritten as

$$\sigma_{\mathbb{R}^3}(\Phi^0, F) = -\mathrm{i}[a_0^*(K_0 F) - a_0(K_0 F)]$$

where $K_0 : [D(K_0) \cap \mathcal{H}_0(\mathbb{R}^3, \mathbb{R})] \to L^2(\mathbb{R}^3)$ is defined in (2.14). The field at any other time is given by

$$\sigma_{\mathbb{R}^3}(\Phi_t^0, F) = \sigma_{\mathbb{R}^3}(\Phi_0, U_0(0, t)F)$$

and therefore, the time evolution mimics the free evolution, $\Phi_t^0 = U_0(t, 0)\Phi^0$.

The time-zero interacting field operator and its conjugate momentum are given by

$$\phi(f) \equiv \frac{1}{\sqrt{2}}[a^*(f) + a(f)]$$

and

$$\pi(g) \equiv \frac{\mathrm{i}}{\sqrt{2}}[a^*(g) - a(g)]$$

where the test functions are real-valued. This choice of ϕ and π yield a representation of the CCRs,

$$[\phi(f), \pi(g)] = \mathrm{i}\langle f, g\rangle_{L^2(\Omega)}$$

although it does not reduce to the standard representation when $\Omega = \mathbb{R}^3$. We choose this representation for both technical reasons, as well as for its simplicity. We introduce the two-component field $\Phi(\cdot)$,

$$\sigma_\Omega(\Phi, F) \equiv \phi(f_2) - \pi(f_1) \tag{3.8}$$

for real-valued $F \in \mathcal{H}(\Omega, \mathbb{R}) = \mathring{\mathcal{H}}_1(\Omega, \mathbb{R}) \oplus L^2(\Omega, \mathbb{R})$. This defines the field at time-zero. At any other time it is defined by

$$\sigma_\Omega(\Phi_t, F) \equiv \sigma_\Omega(\Phi, U(0, t)F) \tag{3.9}$$

which gives $\Phi_t = U(t, 0)\Phi$ just as in the classical problem. Since $U(t, 0)$ preserves the real-valued nature of the data, this definition makes sense. We recover the single-component fields as follows:

$$\phi(f, t) \equiv \sigma_\Omega\left(\Phi, U(0, t)\begin{pmatrix} 0 \\ f \end{pmatrix}\right)$$

and

$$\pi(g, t) \equiv -\sigma_\Omega\left(\Phi, U(0, t)\begin{pmatrix} g \\ 0 \end{pmatrix}\right).$$

It follows that

$$[\sigma_\Omega(\Phi_t, F), \sigma_\Omega(\Phi_t, G)] = \mathrm{i}\sigma_\Omega(U(0,t)F, U(0,t)G) \\ = \mathrm{i}\sigma_\Omega(F, G) \tag{3.10}$$

for real-valued F and G.

Finally, it is important to note that spatial derivatives of the field operators are defined in a distributional sense, e.g.

$$(-\triangle_{\mathrm{D}}^{\Omega})\phi(f,t) = \phi(-\triangle_{\mathrm{D}}^{\Omega} f, t).$$

However, time derivatives are taken to be the strong derivatives of an operator-valued function, e.g.

$$\frac{\mathrm{d}}{\mathrm{d}t}[\phi(f,t)\Psi]$$

for $\Psi \in \mathcal{F}^0_{\mathrm{scatt}}$ (finite particle vectors). It is well known that

$$\|a(f)\Psi\|_{\mathcal{F}_{\mathrm{scatt}}} \leqslant \|f\|_{L^2(\Omega)} \|(\mathbb{N}+1)^{\frac{1}{2}}\Psi\|_{\mathcal{F}_{\mathrm{scatt}}} \tag{3.11}$$

where $\mathbb{N}$ is the number operator, with a similar relation holding for $a^*(\cdot)$ [31]. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\phi(f(t))\Psi] = \phi\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right)\Psi \tag{3.12}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} f(t)$ is the strong derivative of $f(t)$ with respect to $\|\cdot\|_{L^2(\Omega)}$. Similar relations hold for $\pi(f(t))$, as well as for the corresponding free fields. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t) \overset{\|\cdot\|_{\mathcal{H}(\Omega)}}{=} -\mathrm{i}HF(t)$$

where $F(t) = U(t,0)F$, and since $\|\cdot\|^2_{L^2(\Omega)} \oplus \|\cdot\|^2_{L^2(\Omega)} \leqslant c\|\cdot\|^2_{\mathcal{H}(\Omega)}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f_i(t) \overset{\|\cdot\|_{L^2(\Omega)}}{=} [-\mathrm{i}HF(t)]_i \tag{3.13}$$

where $f_i(t)$ and $[-\mathrm{i}HF(t)]_i$ are the ith component of $F(t)$ and $-\mathrm{i}HF(t)$, respectively.

We are finally ready to prove

Theorem 3.1. The fields $\phi(\cdot, t)$ and $\pi(\cdot, t)$ satisfy the following quantum requirements

(1) $\partial_t \phi(\cdot, t) = \pi(\cdot, t)$.

(2) $\phi(\cdot, t)$ satisfies the field equation

$$\partial_t^2 \phi(f,t) - \triangle_{\mathrm{D}}^{\Omega}\phi(f,t) + m^2\phi(f,t) = 0.$$

(3) The fields satisfy the canonical commutation relations

(i) $[\phi(f,t), \phi(g,t)] = 0$

(ii) $[\pi(f,t), \pi(g,t)] = 0$

(iii) $[\phi(f,t), \pi(g,t)] = \mathrm{i}\langle f, g\rangle_{L^2(\Omega)}$.

(4) $\phi(\cdot, t)$ satisfies the principle of microscopic causality.

Proof.

(1) Apply (3.12) and (3.13) to obtain

$$\begin{aligned}\partial_t \phi(f,t) &= \sigma_\Omega\left(\Phi, \partial_t U(0,t)\begin{pmatrix}0\\ f\end{pmatrix}\right)\\ &= \sigma_\Omega\left(\Phi, U(0,t)\mathrm{i}H\begin{pmatrix}0\\ f\end{pmatrix}\right)\\ &= -\sigma_\Omega\left(\Phi_t, \begin{pmatrix}f\\ 0\end{pmatrix}\right)\\ &= \pi(f,t).\end{aligned}$$

(2) Again, from repeated application of (3.12) etc, we have,

$$\begin{aligned}\partial_t^2\phi(f,t) &= \sigma_\Omega\left(\Phi, \partial_t^2 U(0,t)\begin{pmatrix}0\\ f\end{pmatrix}\right)\\ &= \sigma_\Omega\left(\Phi, -U(0,t)H^2\begin{pmatrix}0\\ f\end{pmatrix}\right)\\ &= \sigma_\Omega\left(\Phi_t, \begin{pmatrix}0\\ -(-\triangle_{\mathrm{D}}^{\Omega}+m^2)f\end{pmatrix}\right)\\ &= -(-\triangle_{\mathrm{D}}^{\Omega}+m^2)\phi(f,t).\end{aligned}$$

(3) The CCRs are recovered from (3.10) via a judicious choice of test function, e.g.

$$\begin{aligned}[\phi(f,t),\phi(g,t)] &= \mathrm{i}\sigma_\Omega\left(\begin{pmatrix}0\\ f\end{pmatrix}, \begin{pmatrix}0\\ g\end{pmatrix}\right)\\ &= 0.\end{aligned}$$

The remaining relations are obtained in a similar fashion.

(4) We show that

$$[\phi(f,t),\phi(g,t')] = 0$$

when $B_f(t) \equiv \{(x,s) : x \in \operatorname{supp} f, s = t\}$ and $B_g(t')$ are space-like separated. Consider,

$$\begin{aligned}[\phi(f,t),\phi(g,t')] &= \mathrm{i}\sigma_\Omega\left(U(0,t)\begin{pmatrix}0\\ f\end{pmatrix}, U(0,t')\begin{pmatrix}0\\ g\end{pmatrix}\right)\\ &= \mathrm{i}\sigma_\Omega\left(U(t',t)\begin{pmatrix}0\\ f\end{pmatrix}, \begin{pmatrix}0\\ g\end{pmatrix}\right)\\ &= \mathrm{i}\int_\Omega f_1(t'-t)g\,\mathrm{d}x \qquad (3.14)\end{aligned}$$

where $f_1(t'-t)$ is the first component of $U(t',t)\begin{pmatrix}0\\ f\end{pmatrix}$. Since $B_f(t)$ and $B_g(t')$ are space-like separated, $\operatorname{dist}(y, B_f(0)) > |t'-t|$for $y \in B_g(0)$. However, from theorem A.1, we have $\operatorname{supp} f_1(t'-t) = \{x : \operatorname{dist}(x, B_f(0)) \leqslant |t'-t|\}$. Hence $\operatorname{supp} f(t'-t) \cap \operatorname{supp} g = \varphi$ (the empty set) and (3.14) is zero. □

3.1. Asymptotic fields

We expect the interacting field to behave like a free field in the distant past and future. To sharpen this concept we define time-zero asymptotic fields $\Phi^\pm$,

$$\sigma_{\mathbb{R}^3}(\Phi^\pm, F) \equiv \sigma_\Omega(\Phi, W_\pm F) \qquad (3.15)$$

for $F \in \mathcal{H}_0(\mathbb{R}^3,\mathbb{R})$. Recall from (2.24) that the wave operators are real; therefore (3.15) makes sense. Also, note that $\Phi^\pm$ are operators on $\mathcal{F}_{\mathrm{scatt}}$ even though the test functions are elements of the free Hilbert space. The time evolution mimics the free classical problem,

$$\sigma_{\mathbb{R}^3}(\Phi_t^\pm, F) = \sigma_{\mathbb{R}^3}(\Phi^\pm, U_0(0,t)F)$$

so that formally $\Phi_t^\pm = U_0(t,0)\Phi^\pm$. The single-component fields are recovered as follows,

$$\phi^\pm(f,t) \equiv \sigma_{\mathbb{R}^3}\left(\Phi_t^\pm, \begin{pmatrix}0\\ f\end{pmatrix}\right)$$

and

$$\pi^{\pm}(g,t) \equiv -\sigma_{\mathbb{R}^3}\left(\Phi_t^{\pm}, \begin{pmatrix} g \\ 0 \end{pmatrix}\right).$$

These fields satisfy the free-field quantum requirements. To show this we make use of the following results:

Lemma 1. The wave operators $W_{\pm}$ are symplectic, i.e.

$$\sigma_{\mathbb{R}^3}(F,G) = \sigma_{\Omega}(W_{\pm}F, W_{\pm}G)$$

where $F, G \in \mathcal{H}_0(\mathbb{R}^3, \mathbb{R})$.

Proof. Let $F, G \in C_0^{\infty}(\mathbb{R}^3, \mathbb{R}) \times C_0^{\infty}(\mathbb{R}^3, \mathbb{R})$, and consider,

$$\begin{aligned}\sigma_{\mathbb{R}^3}(F,G) &= \sigma_{\mathbb{R}^3}(U_0(t,0)F, U_0(t,0)G)\\ &= \sigma_{\Omega}(U_0(t,0)F, U_0(t,0)G) + \sigma_{\Lambda}(U_0(t,0)F, U_0(t,0)G).\end{aligned} \tag{3.16}$$

The first term on the right-hand side can be rewritten as

$$\sigma_{\Omega}(U_0(t,0)F, U_0(t,0)G) = \sigma_{\Omega}((I-J^2)U_0(t,0)F, U_0(t,0)G) + \sigma_{\Omega}(W(t)F, W(t)G) \tag{3.17}$$

where in the last term we have used the fact that $\sigma_{\Omega}(\cdot,\cdot)$ is invariant with respect to $U(t,0)$. Notice that the last term in (3.16) and the $(I-J^2)$ term in (3.17) reduce to integrals over bounded sets. However, these terms vanish as $t \to \pm\infty$, since free solutions decay to zero over such sets as shown in (2.9). Therefore,

$$\begin{aligned}\sigma_{\mathbb{R}^3}(F,G) &= \lim_{t\to\pm\infty} \sigma_{\Omega}(W(t)F, W(t)G)\\ &= \sigma_{\Omega}(W_{\pm}F, W_{\pm}G)\end{aligned} \tag{3.18}$$

where in the last step we have made use of (2.18) and (2.22). Finally, from (2.25) and the fact that $C_0^{\infty}(\mathbb{R}^3, \mathbb{R}) \times C^{\infty}(\mathbb{R}^3, \mathbb{R})$ is dense in $\mathcal{H}_0(\mathbb{R}^3, \mathbb{R})$, we have that (3.18) holds on $\mathcal{H}_0(\mathbb{R}^3, \mathbb{R})$. □

Lemma 2. The fields $\Phi_t^{\pm}$ satisfy the following commutation relation

$$[\sigma_{\mathbb{R}^3}(\Phi_t^{\pm}, F), \sigma_{\mathbb{R}^3}(\Phi_t^{\pm}, G)] = \mathrm{i}\sigma_{\mathbb{R}^3}(F,G).$$

Proof. Consider,

$$\begin{aligned}[\sigma_{\mathbb{R}^3}(\Phi_t^{\pm}, F), \sigma_{\mathbb{R}^3}(\Phi_t^{\pm}, G)] &= [\sigma_{\Omega}(\Phi, W_{\pm}U_0(0,t)F), \sigma_{\Omega}(\Phi, W_{\pm}U_0(0,t)G)],\\ &= \mathrm{i}\sigma_{\Omega}(W_{\pm}U_0(0,t)F, W_{\pm}U_0(0,t)G)\\ &= \mathrm{i}\sigma_{\mathbb{R}^3}(U_0(0,t)F, U_0(0,t)G)\\ &= \mathrm{i}\sigma_{\mathbb{R}^3}(F,G)\end{aligned}$$

where in the last two steps we have made use of lemma 1, and the fact that $U_0(t,0)$ is symplectic, respectively. □

We are finally ready to prove,

Theorem 3.2. The fields $\phi^{\pm}(\cdot,t)$ and $\pi^{\pm}(\cdot,t)$ satisfy the following free-field quantum requirements:

(1) $\partial_t \phi^{\pm}(\cdot,t) = \pi^{\pm}(\cdot,t)$.

(2) $\phi^{\pm}(\cdot,t)$ satisfies the free-field equation,

$$\partial_t^2 \phi^{\pm}(f,t) - \triangle\phi^{\pm}(f,t) + m^2\phi^{\pm}(f,t) = 0.$$

(3) The asymptotic fields satisfy the free-field CCRs,

(i) $[\phi^{\pm}(f,t),\phi^{\pm}(g,t)]=0$.
(ii) $[\pi^{\pm}(f,t),\pi^{\pm}(g,t)]=0$.
(iii) $[\phi^{\pm}(f,t),\pi^{\pm}(g,t)]=\mathrm{i}\langle f,g\rangle_{L^2(\mathbb{R}^3)}$.
(4) $\phi^{\pm}(\cdot,t)$ satisfies the principle of microscopic causality.
(5) The scattered field converges to the asymptotic fields in the distant past and future, i.e.

$$\lim_{t\to\pm\infty}\|\{\sigma_\Omega(\Phi_t,JU_0(t,0)F)-\sigma_{\mathbb{R}^3}(\Phi_t^{\pm},U_0(t,0)F)\}\Psi\|_{\mathcal{F}_{\mathrm{scatt}}}=0$$

for any finite particle vector Ψ.

Proof.
(1) We apply the intertwining relations (2.26), (2.27), and obtain,

$$\begin{aligned}\partial_t\phi^{\pm}(f,t)&=\sigma_\Omega\left(\Phi,\partial_t W_{\pm}U_0(0,t)\begin{pmatrix}0\\f\end{pmatrix}\right)\\&=\sigma_\Omega\left(\Phi,\partial_t U(0,t)W_{\pm}\begin{pmatrix}0\\f\end{pmatrix}\right)\\&=\sigma_\Omega\left(\Phi,U(0,t)iHW_{\pm}\begin{pmatrix}0\\f\end{pmatrix}\right)\\&=-\sigma_\Omega\left(\Phi,W_{\pm}U_0(0,t)\begin{pmatrix}f\\0\end{pmatrix}\right)\\&=\pi^{\pm}(f,t).\end{aligned}$$

(2) Again, repeated application of (2.26), (2.27) yields

$$\begin{aligned}\partial_t^2\phi^{\pm}(f,t)&=\sigma_\Omega\left(\Phi,-W_{\pm}U_0(0,t)H_0^2\begin{pmatrix}0\\f\end{pmatrix}\right)\\&=\sigma_{\mathbb{R}^3}\left(\Phi_t^{\pm},\begin{pmatrix}0\\(-\triangle+m^2)f\end{pmatrix}\right)\\&=-(-\triangle+m^2)\phi^{\pm}(f,t).\end{aligned}$$

(3) The CCRs are verified using the results of lemma 2, e.g.

$$\begin{aligned}[\phi^{\pm}(f,t),\pi^{\pm}(g,t)]&=\left[\sigma_{\mathbb{R}^3}\left(\Phi_t^{\pm},\begin{pmatrix}0\\f\end{pmatrix}\right),\sigma_{\mathbb{R}^3}\left(\Phi_t^{\pm},\begin{pmatrix}g\\0\end{pmatrix}\right)\right]\\&=\mathrm{i}\sigma_{\mathbb{R}^3}\left(\begin{pmatrix}0\\f\end{pmatrix},\begin{pmatrix}g\\0\end{pmatrix}\right)\\&=\mathrm{i}\langle f,g\rangle_{L^2(\mathbb{R}^3)}.\end{aligned}$$

The other relations are obtained in a similar fashion.
(4) Just as in theorem 3.1, we show that

$$[\phi^{\pm}(f,t),\phi^{\pm}(g,t')]=0$$

when $B_f(t)$ and $B_g(t')$ are space-like separated. We have

$$\begin{aligned}[\phi^{\pm}(f,t),\phi^{\pm}(g,t')]&=\mathrm{i}\sigma_{\mathbb{R}^3}\left(U_0(0,t)\begin{pmatrix}0\\f\end{pmatrix},U_0(0,t')\begin{pmatrix}0\\g\end{pmatrix}\right)\\&=\mathrm{i}\sigma_{\mathbb{R}^3}\left(U_0(t',t)\begin{pmatrix}0\\f\end{pmatrix},\begin{pmatrix}0\\g\end{pmatrix}\right)\\&=\mathrm{i}\int_{\mathbb{R}^3}f_1(t'-t)g\,\mathrm{d}x\end{aligned}\tag{3.19}$$

where $f_1(t'-t)$ is the first component of $U_0(t',t)\begin{pmatrix}0\\ f\end{pmatrix}$. Since $B_f(t)$ and $B_g(t')$ are space-like separated we have $\mathrm{dist}(y, B_g(0)) > |t'-t|$ for $y \in B_g(0)$. However, it is well known that $\mathrm{supp}\, f(t'-t) = \{x : \mathrm{dist}(x, B_f(0)) \leqslant |t'-t|\}$, and therefore, $\mathrm{supp}\, f(t'-t) \cap \mathrm{supp}\, g = \varphi$ and (3.19) is zero.

(5) Let $\Psi \in \mathcal{F}^0_{\mathrm{scatt}}$ and consider,

$$\begin{aligned}
\lim_{t\to\pm\infty} &\|\{\sigma_\Omega(\Phi_t, JU_0(t,0)F) - \sigma_{\mathbb{R}^3}(\Phi_t^\pm, U_0(t,0)F)\}\Psi\|_{\mathcal{F}_{\mathrm{scatt}}} \\
&= \lim_{t\to\pm\infty} \|\{\sigma_\Omega(\Phi, U(0,t)JU_0(t,0)F) - \sigma_{\mathbb{R}^3}(\Phi, W_\pm F)\}\Psi\|_{\mathcal{F}_{\mathrm{scatt}}} \\
&= \lim_{t\to\pm\infty} \|\{\sigma_\Omega(\Phi, (W(t) - W_\pm)F)\}\Psi\|_{\mathcal{F}_{\mathrm{scatt}}} \\
&\leqslant C_\Psi \lim_{t\to\pm\infty} \|(W(t) - W_\pm)F\|_{\mathcal{H}(\Omega,\mathbb{R})} \\
&= 0
\end{aligned}$$

where C_Ψ is a constant that depends on Ψ. Note, in the second last step we have made use of (3.11). □

3.2. Asymptotic vacuum states

At this point, we have representations for the ‘in’ and ‘out’ asymptotic fields on the same Fock space. However, the goal is to determine scattering amplitudes, and for this we need to compare multiparticle ‘in’ and ‘out’ vectors. These vectors are obtained as follows. We first construct Weyl algebras for the ‘in’ and ‘out’ fields, and then use an algebraic argument to obtain ‘in’ and ‘out’ vacuum states. Next, we use the GNS construction to obtain Hilbert space representations for these states. We then use the classical scattering operator to obtain a representation of the ‘out’ algebra on the ‘in’ Hilbert space, thereby defining a map between the ‘in’ and ‘out’ representations. Finally, we show that this map is unitarily implementable.

To begin the analysis, recall that the operators $\Phi^\pm$ and Φ^0 are self-adjoint on $\mathcal{F}_{\mathrm{scatt}}$ and $\mathcal{F}_{\mathrm{free}}$, respectively. Therefore, from Stone’s theorem, we have unitary Weyl operators,

$$\mathcal{W}_{\mathrm{in}}(F) \equiv \mathrm{e}^{\mathrm{i}\sigma_{\mathbb{R}^3}(\Phi^-,F)} \tag{3.20}$$

$$\mathcal{W}_{\mathrm{out}}(F) \equiv \mathrm{e}^{\mathrm{i}\sigma_{\mathbb{R}^3}(\Phi^+,F)} \tag{3.21}$$

on $\mathcal{F}_{\mathrm{scatt}}$, and

$$\mathcal{W}_0(F) \equiv \mathrm{e}^{\mathrm{i}\sigma_{\mathbb{R}^3}(\Phi^0,F)} \tag{3.22}$$

on $\mathcal{F}_{\mathrm{free}}$. Each of the $\mathcal{W}$s satisfies the Weyl form the CCRs.

$$\mathcal{W}(F_1)\mathcal{W}(F_2) = \mathrm{e}^{-\mathrm{i}\sigma_{\mathbb{R}^3}(F_1,F_2)}\mathcal{W}(F_1+F_2) \tag{3.23}$$

and the fields are recovered from the $\mathcal{W}$s via differentiation, e.g.

$$\sigma_{\mathbb{R}^3}(\Phi^-, F) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{W}_{\mathrm{in}}(\alpha F)|_{\alpha=0}.$$

At this point we digress to review some well known algebraic facts. To begin with, each of the operators (3.20)–(3.22) generates its own C^*-algebra $\mathcal{A}_{\mathrm{in}}$, $\mathcal{A}_{\mathrm{out}}$ and $\mathcal{A}_0$, respectively. These algebras are constructed by first obtaining the set of all finite sums of the form

$$\sum_\alpha c_\alpha \mathcal{W}(F_\alpha) \qquad c_\alpha \in \mathbb{C}$$

and then taking the norm closure of this set in the Banach space of all bounded operators on the representation space. We use the triplet $(\mathcal{A}, \mathcal{W}, \mathcal{H})$ to denote the algebra $\mathcal{A}$, its

Weyl generator $\mathcal{W}$, and the Hilbert space $\mathcal{H}$ on which it acts. We have three such triplets $(\mathcal{A}_{\rm in}, \mathcal{W}_{\rm in}, \mathcal{F}_{\rm scatt})$, $(\mathcal{A}_{\rm out}, \mathcal{W}_{\rm out}, \mathcal{F}_{\rm scatt})$ and $(\mathcal{A}_0, \mathcal{W}_0, \mathcal{F}_{\rm free})$. Moreover $\mathcal{A}_{\rm in}$, $\mathcal{A}_{\rm out}$ and $\mathcal{A}_0$ are isomorphic to one another. This follows from a general equivalence result which states that any two C^*-algebras generated by Weyl operators satisfying (3.23), and defined via the same symplectic form over the same pre-Hilbert space (in this case $\mathcal{H}_0(\mathbb{R}^3, \mathbb{R})$) are isomorphic [31]. We denote these isomorphisms by

$$\gamma_{\rm in/out}[\mathcal{W}_0(F)] = \mathcal{W}_{\rm in/out}(F) \tag{3.24}$$

and

$$s[\mathcal{W}_{\rm in}(F)] = \mathcal{W}_{\rm out}(F). \tag{3.25}$$

We use (3.24) to construct the 'in' and 'out' vacuum states. Finally, note that the time development of the time-zero operators $\mathcal{W}_{\rm in}(\cdot)$,$\mathcal{W}_{\rm out}(\cdot)$ and $\mathcal{W}_0(\cdot)$ is governed by the free dynamics, e.g. $\mathcal{W}_{0,t}(F) \equiv \mathcal{W}_0(U_0(0,t)F)$. Since $U_0(0,t)$ is real, symplectic and invertible it defines a time evolution automorphism $\alpha_0(t,0)\mathcal{W}_0(F) \equiv \mathcal{W}_{0,t}(F)$ with similar relations holding for $\mathcal{W}_{\rm in}(\cdot)$ and $\mathcal{W}_{\rm out}(\cdot)$.

Recall that a state over a C^* algebra is a positive linear functional with norm one. The vacuum state $\omega_0(\cdot)$ over $\mathcal{A}_{\rm out}$ is given by

$$\omega_0(\mathcal{W}_0(F)) \equiv \mathrm{e}^{-\frac{1}{2}\|K_0F\|^2_{L_2(\mathbb{R}^3)}}$$

where K_0 is defined in (2.14). It is well known that $\omega_0(\cdot)$ is stationary under $\alpha_0(t,0)$, as well as automorphisms that represent the full Poincaré group. Moreover, $\alpha_0(t,0)$ is generated by a unitary group with positive energy. Therefore $\omega_0(\cdot)$ is interpreted as a no-particle state and is realized as follows,

$$\omega_0(\cdot) = \langle\Psi_{0,\rm vac}, [\cdot]\Psi_{0,\rm vac}\rangle_{\mathcal{F}_{\rm free}}$$

where $\Psi_{0,\rm vac} = (1, 0, 0, \ldots)$. Similarly, a single-particle state $\omega_{0,f}(\cdot)$ is represented by the vector $\Psi_f = (0, f, 0, \ldots)$,

$$\omega_{0,f}(\cdot) = \langle\Psi_{0,f}, [\cdot]\Psi_{0,f}\rangle_{\mathcal{F}_{\rm free}}$$

and so on for multiparticle states.

We make use of (3.24) to obtain vacuum states for $\mathcal{A}_{\rm in}$ and $\mathcal{A}_{\rm out}$, i.e.

$$\omega_{\rm in,0}(\cdot) \equiv \omega_0 \circ \gamma_{\rm in}^{-1}(\cdot) \tag{3.26}$$

and

$$\omega_{\rm out,0}(\cdot) \equiv \omega_0 \circ \gamma_{\rm out}^{-1}(\cdot). \tag{3.27}$$

One readily checks that

$$\begin{aligned}\omega_{\rm in,0}(\mathcal{W}_{\rm in}(F)) &\equiv \omega_0 \circ \gamma_{\rm in}^{-1}(\mathcal{W}_{\rm in}(F)) \\ &= \omega_0(\mathcal{W}_0(F)) \\ &= \mathrm{e}^{-\frac{1}{2}\|K_0F\|^2_{L_2(\mathbb{R}^3)}}\end{aligned}$$

with a similar calculation applying for $\omega_{\rm out,0}(\cdot)$. We interpret $\omega_{\rm in,0}(\cdot)$ and $\omega_{\rm out,0}$ as no-particle states in the distant past and future, respectively. Similarly, we interpret $\omega_{{\rm in},f}(\cdot) \equiv \omega_{0,f} \circ \gamma_{\rm in}^{-1}(\cdot)$ and $\omega_{{\rm out},f}(\cdot) \equiv \omega_{0,f} \circ \gamma_{\rm out}^{-1}(\cdot)$ as containing a single asymptotic particle in the distant past and future, respectively, and so on for multiparticle states.

We are finally ready to construct the 'in' and 'out' vacuum vectors. For this we use the GNS construction [29]. The essential result is that given a state $\omega(\cdot)$ over a C^* algebra

$\mathcal{A}$ one can define a cyclic representation $\tilde{\mathcal{A}}$, of $\mathcal{A}$, acting in a Hilbert space $\tilde{\mathcal{H}}$ with cyclic vector Ψ_ω such that

$$\omega(\mathcal{A}) = \langle \Psi_\omega, \tilde{\mathcal{A}}\Psi_\omega \rangle_{\tilde{\mathcal{H}}}.$$

We apply this result to states $\omega_{\text{in},0}$ and $\omega_{\text{out},0}$ and obtain new representations ($\tilde{\mathcal{A}}_{\text{in}}$, $\tilde{\mathcal{W}}_{\text{in}}$, $\tilde{\mathcal{H}}_{\text{in}}$) and ($\hat{\mathcal{A}}_{\text{out}}$, $\hat{\mathcal{W}}_{\text{out}}$,$\hat{\mathcal{H}}_{\text{out}}$) with vacuum vectors $\tilde{\Psi}_{\text{in,vac}}$ and $\hat{\Psi}_{\text{out,vac}}$, respectively. These new representations are unitarily equivalent to ($\mathcal{A}_0$, $\mathcal{W}_0$, $\mathcal{F}_{\text{free}}$) [32]. Specifically, we have $U_{\text{in}} : \tilde{\mathcal{H}}_{\text{in}} \to \mathcal{F}_{\text{free}}$, where

$$U_{\text{in}}\tilde{\mathcal{W}}_{\text{in}}(F)\tilde{\Psi}_{\text{in,vac}} \equiv \mathcal{W}_0(F)\Psi_{0,\text{vac}}. \tag{3.28}$$

This operator provides a means of identifying particle vectors in $\tilde{\mathcal{H}}_{\text{in}}$. For example, the vectors $\tilde{\Psi}_{\text{in,vac}} = U_{\text{in}}^{-1}\Psi_{0,\text{vac}}$, and $\tilde{\Psi}_{\text{in},f} = U_{\text{in}}^{-1}\Psi_{0,f}$ are no-particle and single-particle vectors, respectively, and so on for multiparticle vectors. A similar analysis applies to ($\hat{\mathcal{A}}_{\text{out}}$, $\hat{\mathcal{W}}_{\text{out}}$, $\hat{\mathcal{H}}_{\text{out}}$), with a corresponding definition for U_{out}.

3.3. The scattering operator

We are finally ready to construct the scattering operator. For this, we follow closely the work of Dimock and Kay [10]. We choose to work with the representation ($\tilde{\mathcal{A}}_{\text{in}}$, $\tilde{\mathcal{W}}_{\text{in}}$, $\tilde{\mathcal{H}}_{\text{in}}$). The first step is to obtain a representation of $\mathcal{W}_{\text{out}}$ on $\tilde{\mathcal{H}}_{\text{in}}$. From (3.15) we have

$$\sigma_{\mathbb{R}^3}(\Phi^+, F) = \sigma_{\mathbb{R}^3}(\Phi^-, S_{\text{cl}}^{-1}F)$$

from which we define

$$\tilde{\mathcal{W}}_{\text{out}}(F) \equiv \tilde{\mathcal{W}}_{\text{in}}(S_{\text{cl}}^{-1}F)$$

which is the desired representation. This defines a mapping,

$$\mathfrak{S}\tilde{\mathcal{W}}_{\text{in}}(F) = \tilde{\mathcal{W}}_{\text{out}}(F). \tag{3.29}$$

It remains to be shown that (3.29) is unitarily implementable, i.e. that there exists a unitary operator on $\tilde{\mathcal{H}}_{\text{in}}$ such that

$$\tilde{\mathcal{S}}\tilde{\mathcal{W}}_{\text{in}}(F)\tilde{\mathcal{S}}^{-1} = \tilde{\mathcal{W}}_{\text{out}}(F). \tag{3.30}$$

It should be noted that the operator $\tilde{\mathcal{S}}$ that we use is the inverse of the scattering operator as defined in some texts [26]. Assuming such an operator exists, then the out vacuum (on $\tilde{\mathcal{H}}_{\text{in}}$) is given by

$$\tilde{\Psi}_{\text{out,vac}} = \tilde{\mathcal{S}}\tilde{\Psi}_{\text{in,vac}}.$$

To see this, consider,

$$\begin{aligned}
\langle \tilde{\Psi}_{\text{out,vac}}, \tilde{\mathcal{W}}_{\text{out}}(F)\tilde{\Psi}_{\text{out,vac}} \rangle_{\tilde{\mathcal{H}}_{\text{in}}} &= \langle \tilde{\mathcal{S}}\tilde{\Psi}_{\text{in,vac}}, \tilde{\mathcal{W}}_{\text{out}}(F)\tilde{\mathcal{S}}\tilde{\Psi}_{\text{in,vac}} \rangle_{\tilde{\mathcal{H}}_{\text{in}}} \\
&= \langle \tilde{\Psi}_{\text{in,vac}}, \tilde{\mathcal{S}}^{-1}\tilde{\mathcal{W}}_{\text{out}}(F)\tilde{\mathcal{S}}\tilde{\Psi}_{\text{in,vac}} \rangle_{\tilde{\mathcal{H}}_{\text{in}}} \\
&= \langle \tilde{\Psi}_{\text{in,vac}}, \tilde{\mathcal{W}}_{\text{in}}(F)\tilde{\Psi}_{\text{in,vac}} \rangle_{\tilde{\mathcal{H}}_{\text{in}}} \\
&= \langle \tilde{\Psi}_{0,\text{vac}}, U_{\text{in}}\tilde{\mathcal{W}}_{\text{in}}(F)U_{\text{in}}^{-1}\tilde{\Psi}_{0,\text{vac}} \rangle_{\mathcal{F}\text{free}} \\
&= \mathrm{e}^{-\frac{1}{2}\|K_0F\|^2_{L_2(\mathbb{R}^3)}}.
\end{aligned}$$

Also, if such an operator exits, we can use it to compute transition amplitudes between multiparticle ‘in’ and ‘out’ states, i.e.

$$\begin{aligned}
S_{f_1\ldots f_N, g_1\ldots g_M} &\equiv \langle \tilde{\Psi}_{\text{in},f_1\ldots f_N}, \tilde{\mathcal{S}}\tilde{\Psi}_{\text{in},g_1\ldots g_M} \rangle_{\tilde{\mathcal{H}}_{\text{in}}} \\
&= \langle \tilde{\Psi}_{\text{in},f_1\ldots f_N}, \tilde{\Psi}_{\text{out},g_1\ldots g_M} \rangle_{\tilde{\mathcal{H}}_{\text{in}}}.
\end{aligned} \tag{3.31}$$

The amplitude (3.31) is the probability of obtaining the state $\tilde{\Psi}_{\mathrm{out},g_1\ldots g_M}$ in the distant future given that the system was in the state $\tilde{\Psi}_{\mathrm{in},f_1\ldots f_N}$ in the distant past.

Before proving the implementability of (3.29) we review some needed results. We start with the concept of a 'one-particle structure' $(K, \mathfrak{H}, \mathrm{e}^{-\mathrm{i}ht})$ for a linear dynamical system $(\mathcal{D}, \sigma(\cdot,\cdot), \mathcal{U}(t,0))$ consisting of a real vector space $\mathcal{D}$, a symplectic form, and a one-parameter symplectic group [2, 10, 33]. The one-particle structure consists of a complex Hilbert space $\mathfrak{H}$ regarded as a real symplectic space with symplectic form $2\,\mathrm{Im}\langle\cdot,\cdot\rangle_{\mathfrak{H}}$, a symplectic operator $K : \mathcal{D} \to \mathfrak{H}$,

$$2\,\mathrm{Im}\langle KF_1, KF_2\rangle_{\mathfrak{H}} = \sigma(F_1, F_2)$$

that maps onto a dense domain, and a unitary group $\mathrm{e}^{-\mathrm{i}ht}$ on $\mathfrak{H}$ with strictly positive generator h satisfying

$$KU(t,0) = \mathrm{e}^{-\mathrm{i}ht}K.$$

An important feature of these structures, which we exploit, is that they are unique in the sense that for any two such structures $(K_1, \mathfrak{H}_1, \mathrm{e}^{-\mathrm{i}h_1t})$ and $(K_2, \mathfrak{H}_2, \mathrm{e}^{-\mathrm{i}h_2t})$, the operator

$$\Sigma = K_1K_2^{-1}$$

which is defined on the dense domain $K_2\mathcal{D}$, extends to a unitary operator $\Sigma : \mathfrak{H}_2 \to \mathfrak{H}_1$ [34].

Now, consider the free dynamical system $(\mathcal{H}_0(\mathbb{R}^3,\mathbb{R}), \sigma_{\mathbb{R}^3}(\cdot,\cdot), U_0(t,0))$. It is well known that $(K_0, L_2(\mathbb{R}^3), \mathrm{e}^{-\mathrm{i}B_0t})$ is a one-particle structure for this system [10]. We also have,

Lemma 3. The triplet $(K_0S_{\mathrm{cl}}, L_2(\mathbb{R}^3), \mathrm{e}^{-\mathrm{i}B_0t})$ is one-particle structure for the free system $(\mathcal{H}_0(\mathbb{R}^3,\mathbb{R}), \sigma_{\mathbb{R}^3}(\cdot,\cdot), U_0(t,0))$.

Proof. First, we show that $K_0S_{\mathrm{cl}} : \mathcal{H}_0(\mathbb{R}^3,\mathbb{R}) \to L_2(\mathbb{R}^3)$ onto a dense set. This follows from the fact that $S_{\mathrm{cl}} : \mathcal{H}_0(\mathbb{R}^3,\mathbb{R}) \to \mathcal{H}_0(\mathbb{R}^3,\mathbb{R})$ is a bijection, and that K_0 maps $\mathcal{H}_0(\mathbb{R}^3,\mathbb{R})$ onto a dense set in $L_2(\mathbb{R}^3)$. It remains to show that

$$K_0S_{\mathrm{cl}}U_0(t,0) = \mathrm{e}^{-\mathrm{i}B_0t}K_0S_{\mathrm{cl}}. \tag{3.32}$$

This follows from (2.29) and the fact that $K_0U_0(t,0) = \mathrm{e}^{-\mathrm{i}B_0t}K_0$. □

At this point we have two one-particle structures $(K_0, L_2(\mathbb{R}^3), \mathrm{e}^{-\mathrm{i}B_0t})$ and $(K_0S_{\mathrm{cl}}, L_2(\mathbb{R}^3), \mathrm{e}^{-\mathrm{i}B_0t})$. From the equivalence result of Kay [34],

$$\Sigma \equiv K_0S_{\mathrm{cl}}K_0^{-1}$$

extends to a unitary operator on $L_2(\mathbb{R}^3)$. We exploit this fact as follows. First, note that $\mathcal{W}_0(F)$ can be written as

$$\mathcal{W}_0(F) = \hat{\mathcal{W}}_0(K_0F) \tag{3.33}$$

where

$$\hat{\mathcal{W}}_0(f) = \mathrm{e}^{\mathrm{i}[a_0(f) - a_0^*(f)]}$$

for $f \in L^2(\mathbb{R}^3)$. This is the Fock representation of the Weyl algebra,

$$\hat{\mathcal{W}}_0(f_1)\hat{\mathcal{W}}_0(f_2) = \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\langle f_1,f_2\rangle_{L^2(\mathbb{R}^3)}}\hat{\mathcal{W}}_0(f_1+f_2)$$

over $(L^2(\mathbb{R}^3), 2\,\mathrm{Im}\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^3)})$ [31]. Now, since Σ is unitary on $L^2(\mathbb{R}^3)$, we have

$$S^{-1}\hat{\mathcal{W}}_0(f)S \equiv \hat{\mathcal{W}}_0(\Sigma f) \tag{3.34}$$

where S is the second quantization of Σ, i.e.

$$S \equiv I \oplus \Sigma \oplus (\Sigma \otimes \Sigma) \oplus (\Sigma \otimes \Sigma \otimes \Sigma) \oplus \cdots .$$

From (3.33) and (3.34) we have

$$\begin{aligned} S\mathcal{W}_0(F)S^{-1} &= S\hat{\mathcal{W}}_0(K_0F)S^{-1} \\ &= \hat{\mathcal{W}}_0(\Sigma^{-1}K_0F) \\ &= \mathcal{W}_0(S_{\mathrm{cl}}^{-1}F) \end{aligned} \tag{3.35}$$

where $\Sigma^{-1} = K_0 S_{\mathrm{cl}}^{-1} K_0^{-1}$. This result is the key to the implementability of (3.29) which we now prove.

Theorem 3.3. The mapping

$$\mathfrak{S}\tilde{\mathcal{W}}_{\mathrm{in}}(F) = \tilde{\mathcal{W}}_{\mathrm{out}}(F)$$

is unitarily implementable.

Proof. From (3.28) we have

$$\mathcal{W}_0(F) = U_{\mathrm{in}}\tilde{\mathcal{W}}_{\mathrm{in}}(F)U_{\mathrm{in}}^{-1}. \tag{3.36}$$

Substituting (3.36) into the left-hand side of (3.35) yields

$$S\mathcal{U}_{\mathrm{in}}\tilde{\mathcal{W}}_{\mathrm{in}}(F)\mathcal{U}_{\mathrm{in}}^{-1}S^{-1} = \mathcal{W}_0(S_{\mathrm{cl}}^{-1}F). \tag{3.37}$$

Next, apply (3.28) to both sides of (3.37) and obtain

$$\begin{aligned} \mathcal{U}_{\mathrm{in}}^{-1}S\mathcal{U}_{\mathrm{in}}\tilde{\mathcal{W}}_{\mathrm{in}}(F)\mathcal{U}_{\mathrm{in}}^{-1}S^{-1}\mathcal{U}_{\mathrm{in}} &= \mathcal{U}_{\mathrm{in}}^{-1}\mathcal{W}_0(S_{\mathrm{cl}}^{-1}F)\mathcal{U}_{\mathrm{in}} \\ &= \tilde{\mathcal{W}}_{\mathrm{in}}(S_{\mathrm{cl}}^{-1}F) \\ &= \tilde{\mathcal{W}}_{\mathrm{out}}(F). \end{aligned} \tag{3.38}$$

Comparing (3.30) with (3.38) we see that

$$\tilde{\mathcal{S}} \equiv \mathcal{U}_{\mathrm{in}}^{-1}S\mathcal{U}_{\mathrm{in}} \tag{3.39}$$

is the desired unitary quantum scattering operator. □

Finally, it is instructive to compute the action of $\tilde{\mathcal{S}}$ on an n-particle in vector. Let $\Psi_0^{(n)} \in \mathcal{F}_{\mathrm{free}}^0$ be an n-particle vector,

$$\Psi_0^{(n)} = (0, 0, \ldots, \Psi^{(n)}, 0 \ldots)$$

where

$$\Psi^{(n)} = \frac{1}{n!}\sum_{\pi} f_{\pi_1} \otimes f_{\pi_2} \otimes \cdots \otimes f_{\pi_n}$$

as defined in (3.3). The corresponding vector in $\tilde{\mathcal{H}}_{\mathrm{in}}$ is $\Psi_{\mathrm{in}}^{(n)} = \mathcal{U}_{\mathrm{in}}^{-1}\Psi_0^{(n)}$. Now, consider the action of $\tilde{\mathcal{S}}$ on this vector,

$$\begin{aligned} \Psi_{\mathrm{out}} &= \tilde{\mathcal{S}}\Psi_{\mathrm{in}}^{(n)} \\ &= \mathcal{U}_{\mathrm{in}}^{-1}S\mathcal{U}_{\mathrm{in}}\mathcal{U}_{\mathrm{in}}^{-1}\Psi_0^{(n)} \\ &= \mathcal{U}_{\mathrm{in}}^{-1}S\Psi_0^{(n)} \\ &= \mathcal{U}_{\mathrm{in}}^{-1}\Psi_{0,S}^{(n)} \end{aligned}$$

where

$$\Psi_{0,S}^{(n)} = (0, 0, \ldots, \Psi_S^{(n)}, 0 \ldots)$$

with

$$\Psi_S^{(n)} = \frac{1}{n!}\sum_{\pi} \Sigma f_{\pi_1} \otimes \Sigma f_{\pi_2} \otimes \cdots \otimes \Sigma f_{\pi_n}.$$

Thus, we see that the n-particle 'in' vector scatters to an n-particle 'out' vector, implying that there is no particle creation/annihilation. This can be understood on a more intuitive level by noting that the classical interacting Hamiltonian can be diagonalized by an operator similar to (2.13), and that the resulting positive and negative frequency components propagate independently of one another, i.e. there is no mixing of these modes.

Acknowledgment

The author would like to thank Professor J Dimock for suggesting this problem and for numerous helpful discussions.

Appendix A

In this section we specify the support properties of smooth solutions of the Klein Gordon equation on exterior domains.

Theorem A.1. Let $\phi(t,\cdot)$ be a smooth solution of the exterior Dirichlet problem for the Klein–Gordon equation. If the initial data $\phi(0,\cdot)$, $\partial_t\phi(0,\cdot)$ have support $\mathcal{O} \subset \Omega$, then the support of $\phi(t,\cdot)$ is contained in

$$M(\mathcal{O},t) \equiv \{x \in \Omega : \mathrm{dist}(x,\mathcal{O}) \leqslant |t|\}.$$

Proof. This proof follows closely the standard proof for the wave equation ($m = 0$) on exterior domains [24]. Specifically, we show that if the initial data are zero on the exterior of $\mathcal{O}$ then the solution will be zero on the exterior of $M(\mathcal{O},t)$. It suffices to show that if the data are zero on a ball

$$B_{x_0}(R) = \{x : |x - x_0| < R\}$$

not intersecting $\mathcal{O}$, then at any later time T the solution will be zero on the ball $B_{x_0}(R-T)$ where $0 \leqslant T \leqslant R$.

We start by multiplying the field equation (2.1) by $\partial_t\phi$ and then integrating the resulting divergence over a truncated spacetime cone region G with bottom and top defined by $B_{x_0}(R)$ and $B_{x_0}(R-T)$, respectively, i.e.

$$\int_G \{\partial_t[(\partial_t\phi)^2 + \nabla\phi\cdot\nabla\phi + m^2\phi^2] + \nabla\cdot(-2\partial_t\phi\nabla\phi)\}\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{A.1}$$

We apply Gauss's theorem and equate (A.1) to an integral over the boundary of G consisting of top $B_{x_0}(R)$, bottom $B_{x_0}(R-T)$, side C_T, and remaining side portion $\partial\Omega_T \equiv ([0,T]\times\partial\Omega)\cap G$ which represents the most general case in which G intersects the obstacle Λ, i.e.

$$\begin{aligned} 0 = E(T) - E(0) &+ \int_{C_T}\{n_0[(\partial_t\phi)^2 + \nabla\phi\cdot\nabla\phi + m^2\phi^2] + \boldsymbol{n}\cdot(-2\partial_t\phi\nabla\phi)\}\,\mathrm{d}a' \\ &+ \int_{\partial\Omega_T}\boldsymbol{n}\cdot(-2\partial_t\phi\nabla\phi)\,\mathrm{d}a \end{aligned} \tag{A.2}$$

where

$$E(T) \equiv \int_{B_{x_0}(R-T)}[(\partial_t\phi)^2 + \nabla\phi\cdot\nabla\phi + m^2\phi^2]|_{t=T}\,\mathrm{d}x$$

and $\mathrm{d}a'$ and $\mathrm{d}a$ are the differential area elements on C_T and $\partial\Omega_T$, respectively, and $\hat{n} = (n_0, \boldsymbol{n})$ is the unit normal vector to the boundary of G.

Since $\phi(t, x)$ is a smooth solution and satisfies that classical Dirichlet condition, the integral over $\partial\Omega_T$ is zero. Moreover, on C_T we have

$$n_0^2 = \boldsymbol{n} \cdot \boldsymbol{n} = \tfrac{1}{2}$$

and therefore, the integral over C_T can be rewritten as

$$\sqrt{2} \int_{C_T} \left\{ \sum_{j=1}^{3} (n_0 \partial_{x_j} \phi - n_j \partial_t \phi)^2 + n_0^2 m^2 \phi^2 \right\} \mathrm{d}a'$$

which is positive definite. Thus from (A.2) we have

$$E(T) \leqslant E(0)$$

which implies that if ϕ is zero on $B_{x_0}(R)$ then it is zero on $B_{x_0}(R - T)$. □

Appendix B

In this section we prove the existence and completeness of the classical wave operators.

Theorem B.1. The wave operators $W_\pm$ defined in (2.22) exist and are complete, i.e.

$$Ran(W_\pm) = P_{\mathrm{ac}}(H)\mathcal{H}(\Omega)$$

where $P_{\mathrm{ac}}(H)$ is the projection onto the absolutely continuous spectrum of H. Moreover, $W_\pm$ are independent of the choice of identification operator.

Proof. We follow closely an existing proof for the wave equation ($m = 0$) on exterior domains [26, theorem XI.78]. The strategy is to reduce the existence and completeness of our two-component wave operators $W_\pm$ to the existence and completeness of related single-component wave operators [26, theorems XI.75 and XI.76].

We begin with some preliminary definitions. Let $-\triangle_{\mathrm{D}}^{\Omega}$ and $-\triangle_{\mathrm{D}}^{\mathring{\Lambda}}$ be the Dirichlet Laplacians on the open sets Ω and $\mathring{\Lambda}$ ($\mathring{\Lambda}$ denotes the interior of Λ) [25]. We define $-\triangle_{\mathrm{D}}^{\Omega\cup\mathring{\Lambda}} \equiv -\triangle_{\mathrm{D}}^{\Omega} \oplus -\triangle_{\mathrm{D}}^{\mathring{\Lambda}}$ which is self-adjoint on $L^2(\Omega \cup \mathring{\Lambda}) = L^2(\Omega) \oplus L^2(\mathring{\Lambda})$ with $D(-\triangle_{\mathrm{D}}^{\Omega\cup\mathring{\Lambda}}) \equiv D(-\triangle_{\mathrm{D}}^{\Omega}) \oplus D(-\triangle_{\mathrm{D}}^{\mathring{\Lambda}})$. Note, $L^2(\Omega \cup \mathring{\Lambda}) \subset L^2(\mathbb{R}^3)$. We further define the Dirichlet Laplacian on $L^2(\mathbb{R}^3)$ with boundary $\partial\Omega$,

$$-\triangle_{\mathrm{D}}^{\mathbb{R}^3\backslash\partial\Omega} = \mathbb{U}(-\triangle_{\mathrm{D}}^{\Omega\cup\mathring{\Lambda}})\mathbb{U}^{-1}$$

where $\mathbb{U} : L^2(\Omega\cup\mathring{\Lambda}) \to L^2(\mathbb{R}^3)$ is the (unitary) natural injection and $\mathbb{U}^{-1} \equiv P_{L^2(\Omega)} \oplus P_{L^2(\mathring{\Lambda})}$ (projection onto closed subspace). It follows that $-\triangle_{\mathrm{D}}^{\mathbb{R}^3\backslash\partial\Omega}$ is self-adjoint on $L^2(\mathbb{R}^3)$.

We extend these concepts to the two-component formalism. Let

$$\mathcal{H}(\Omega \cup \mathring{\Lambda}) \equiv \mathcal{H}(\Omega) \oplus \mathcal{H}(\mathring{\Lambda})$$

where $\mathcal{H}(\Omega) \equiv D(\sqrt{-\triangle_{\mathrm{D}}^{\Omega}}) \oplus L^2(\Omega)$ etc. The elements of $\mathcal{H}(\Omega \cup \mathring{\Lambda})$ are represented by pairs of the form $F = (F_\Omega, F_{\mathring{\Lambda}})$. We define an interacting Hamiltonian $\hat{H} = H_\Omega \oplus H_{\mathring{\Lambda}}$ where

$$H_\Omega = \mathrm{i} \begin{pmatrix} 0 & I \\ -B_\Omega^2 & 0 \end{pmatrix}$$

and $B_\Omega^2 = -\triangle_{\mathrm{D}}^{\Omega} + m^2$ with corresponding definitions for $H_{\overset{\circ}{\Lambda}}$. The operator $\hat{H}$ is self-adjoint on $\mathcal{H}(\Omega \cup \overset{\circ}{\Lambda})$ with

$$D(\hat{H}) = D(H_\Omega) \oplus D(H_{\overset{\circ}{\Lambda}})$$

where

$$D(H_\Omega) = D(-\triangle_{\mathrm{D}}^{\Omega}) \oplus D\left(\sqrt{-\triangle_{\mathrm{D}}^{\Omega}}\right)$$

with similar definitions for $D(H_{\overset{\circ}{\Lambda}})$. From Stone's theorem we have the unitary group $\hat{U}(t) = \mathrm{e}^{-\mathrm{i}\hat{H}t}$ where,

$$\hat{U}(t)F = (\mathrm{e}^{-\mathrm{i}H_\Omega t}F_\Omega, \mathrm{e}^{-\mathrm{i}H_{\overset{\circ}{\Lambda}}t}F_{\overset{\circ}{\Lambda}}).$$

Next, we introduce the notation

$$\hat{W}_\pm(\hat{H}, H_0; J) \equiv s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}\hat{H}t} J \mathrm{e}^{-\mathrm{i}H_0 t}.$$

Recall, $\mathcal{H}_0(\mathbb{R}^3) = \mathcal{H}_1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, and note $\mathcal{H}(\Omega \cup \overset{\circ}{\Lambda}) \subset \mathcal{H}_0(\mathbb{R}^3)$. Let $\mathcal{I} : \mathcal{H}(\Omega \cup \overset{\circ}{\Lambda}) \to \mathcal{H}_0(\mathbb{R}^3)$ denote the natural injection, and then $\mathcal{I}^* : \mathcal{H}_0(\mathbb{R}^3) \to \mathcal{H}(\Omega \cup \overset{\circ}{\Lambda})$ is the projection onto $\mathcal{H}(\Omega \cup \overset{\circ}{\Lambda})$, i.e.

$$\mathcal{I}^* = P_{\mathcal{H}(\Omega\cup\overset{\circ}{\Lambda})}.$$

Now, Reed and Simon show that $\hat{W}(\hat{H}, H_0; \mathcal{I}^*)$ exist and are complete if and only if the wave operators

$$\tilde{W}_\pm \equiv s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}B_0^2 t} \mathrm{e}^{-\mathrm{i}B^2_{\mathbb{R}^3\setminus\partial\Omega} t}$$

exist and are complete on $L^2(\mathbb{R}^3)$ where $B^2_{\mathbb{R}^3\setminus\partial\Omega} = -\triangle_{\mathrm{D}}^{\mathbb{R}^3\setminus\partial\Omega} + m^2$ [26, theorems XI.75 and XI.76]. Moreover, they prove the existence and completeness of $\overset{\circ}{W}_\pm \equiv s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}(-\triangle)t} \mathrm{e}^{-\mathrm{i}(-\triangle_{\mathrm{D}}^{\mathbb{R}^3\setminus\partial\Omega})t}$, i.e. for $m = 0$. However, from the spectral theorem we have

$$\mathrm{e}^{-\mathrm{i}B^2_{\mathbb{R}^3\setminus\partial\Omega} t} = \mathrm{e}^{-\mathrm{i}m^2 t} \mathrm{e}^{-\mathrm{i}(-\triangle_{\mathrm{D}}^{\mathbb{R}^3\setminus\partial\Omega})t}$$

and therefore,

$$\mathrm{e}^{\mathrm{i}B_0^2 t} \mathrm{e}^{-\mathrm{i}B^2_{\Omega\cup\Lambda} t} = \mathrm{e}^{\mathrm{i}(-\triangle)t} \mathrm{e}^{-\mathrm{i}(-\triangle_{\mathrm{D}}^{\mathbb{R}^3\setminus\partial\Omega})t}$$

which implies that $\tilde{W}_\pm$ exist and are complete, i.e. $\tilde{W}_\pm = \overset{\circ}{W}_\pm$. This, in turn, implies the existence and completeness of $\hat{W}_\pm(\hat{H}, H_0; \mathcal{I}^*)$.

Next, we prove that $\hat{W}_\pm(\hat{H}, H_0; J)$ exist and are complete. It suffices to show that $\mathcal{I}^*$ and J are asymptotically H_0-equivalent [26, proposition 5]. For this we show that

$$\lim_{t\to\pm\infty} \|(\mathcal{I}^* - J)\mathrm{e}^{-\mathrm{i}H_0 t}F\|_{\mathcal{H}(\Omega\cup\overset{\circ}{\Lambda})} = 0 \tag{B.1}$$

for $F \in \mathcal{H}_0(\mathbb{R}^3)$. Note that, $\mathcal{H}(\Omega \cup \overset{\circ}{\Lambda}) \subset \mathcal{H}_0(\mathbb{R}^3)$, and therefore, $\|\mathcal{I}^*F\|_{\mathcal{H}(\Omega\cup\overset{\circ}{\Lambda})} \leqslant \|\mathcal{I}^*F\|_{\mathcal{H}_0(\mathbb{R}^3)}$. Consider

$$\|(\mathcal{I}^* - J)\mathrm{e}^{-\mathrm{i}H_0 t}F\|_{\mathcal{H}(\Omega\cup\overset{\circ}{\Lambda})} \leqslant \|(\mathcal{I}^* - I)\mathrm{e}^{-\mathrm{i}H_0 t}F\|_{\mathcal{H}_0(\mathbb{R}^3)} + \|(J - I)\mathrm{e}^{-\mathrm{i}H_0 t}F\|_{\mathcal{H}_0(\mathbb{R}^3)}. \tag{B.2}$$

We use $\mathcal{I}^* - I = P_{\mathcal{H}(\Omega\cup\mathring{\Lambda})^\perp}$ to estimate the first term on the right-hand side of (B.2), i.e.

$$\begin{aligned}\|P_{\mathcal{H}(\Omega\cup\mathring{\Lambda})^\perp}\mathrm{e}^{-\mathrm{i}H_0t}F\|_{\mathcal{H}_0(\mathbb{R}^3)} &\leqslant \|P_{\mathcal{H}(\Omega\cup\mathring{\Lambda})^\perp}J\mathrm{e}^{-\mathrm{i}H_0t}F\|_{\mathcal{H}_0(\mathbb{R}^3)}\\ &\quad+\|P_{\mathcal{H}(\Omega\cup\mathring{\Lambda})^\perp}(I-J)\mathrm{e}^{-\mathrm{i}H_0t}F\|_{\mathcal{H}_0(\mathbb{R}^3)}.\end{aligned} \tag{B.3}$$

The first term on the right-hand side of (B.3) is zero because $J\mathrm{e}^{-\mathrm{i}H_0t}F\in\mathcal{H}(\Omega)$. Therefore, from (B.2) and (B.3) we have

$$\begin{aligned}\|(\mathcal{I}^*-J)\mathrm{e}^{-\mathrm{i}H_0t}F\|_{\mathcal{H}(\Omega\cup\mathring{\Lambda})} &\leqslant \|P_{\mathcal{H}(\Omega\cup\mathring{\Lambda})^\perp}(I-J)\mathrm{e}^{-\mathrm{i}H_0t}F\|_{\mathcal{H}_0(\mathbb{R}^3)}+\|(J-I)\mathrm{e}^{-\mathrm{i}H_0t}F\|_{\mathcal{H}_0(\mathbb{R}^3)}\\ &\leqslant 2\|(J-I)\mathrm{e}^{-\mathrm{i}H_0t}F\|_{\mathcal{H}_0(\mathbb{R}^3)}\end{aligned} \tag{B.4}$$

where in the last step we have used the fact that $I-J:\mathcal{H}_0(\mathbb{R}^3)\to\mathcal{H}(\Omega\cup\mathring{\Lambda})^\perp$. Now, since $J-I$ is zero off a compact set, and since $\mathrm{e}^{-\mathrm{i}H_0t}F$ decays to zero over any such set as shown in (2.9), we have that (B.4) tends to zero in the limit which implies the desired result (B.1).

At this point we have the existence and completeness of $\hat{W}_\pm(\hat{H},H_0;J)$ with

$$Ran(\hat{W}_\pm(\hat{H},H_0;J)) = P_{\mathrm{ac}}(\hat{H})\mathcal{H}(\Omega\cup\mathring{\Lambda}). \tag{B.5}$$

We are finally ready to prove the completeness of our operators $W_\pm$. First, note that

$$J\mathrm{e}^{-\mathrm{i}H_0t}F = (J\mathrm{e}^{-\mathrm{i}H_0t}F, 0)$$

i.e. the component of the subspace $\mathcal{H}(\mathring{\Lambda})$ is zero. Therefore, we have

$$\mathrm{e}^{\mathrm{i}\hat{H}t}J\mathrm{e}^{-\mathrm{i}H_0t}F = (\mathrm{e}^{\mathrm{i}H_\Omega t}J\mathrm{e}^{-\mathrm{i}H_0t}F, 0)$$

which in the limit gives

$$\hat{W}_\pm(\hat{H},H_0;J_{R_0}) = W_\pm\oplus 0 \tag{B.6}$$

which, in turn, implies that

$$Ran(\hat{W}_\pm(\hat{H},H_0;J)) = Ran(W_\pm)\oplus 0. \tag{B.7}$$

From (B.5) and (B.7) we have

$$Ran(W_\pm) = P_{\mathrm{ac}}(H_\Omega)\mathcal{H}(\Omega)$$

and therefore $W_\pm$ are complete. Finally, $W_\pm$ are independent of the choice of identification operator because any two such operators differ only on a bounded set, and the free solutions decay to zero over any such set. □

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